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# DISCRETE PARTITION FUNCTION GAMES

by

William F. Lucas*
and
John C. Maceli**



*Cornell University
**Ithaca College

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## 1. Introduction.

The von Neumann-Morgenstern development in 1944 of a theory for n-person cooperative games leads after considerable argument to a formulation in terms of a real-valued characteristic function which is defined on the set of all subsets of the set of n players. Arguments leading to this formulation as well as their concept of a solution have been challenged on several grounds. Since that time there have been several dozen reformulations or additional models proposed and analyzed to varying degrees. No one of the models proposed is completely satisfactory by itself for all potential applications. However, many of these models when taken together do provide substantial insight, and a particular one does on occasion prove to be quite adequate for analyzing a specific type of application.

Coalition formation and the amount of worth, wealth or power achievable by a coalition is a most crucial aspect in the multiperson cooperative theory. So most models are characterized by assigning a real-numbered value or a set of realizable payoff vectors to each coalition in some manner. One then uses these numbers or sets of vectors to define a set of realizable payoff vectors or imputations. A solution concept then selects some of these payoffs as the ones "most likely" to finally occur, on the basis of some argument, as the resulting distribution to the players in a play of the game.

There are, however, many shortcomings in the more classical theories due to the facts that they assume side payments in some transferable utility, that the actual dynamics of moving from one proposed payoff to another is not explicit, and that they are static in the sense that they do not exhibit the actual negotiations, bargaining, or formation or dissolution of the coalition structures which may take place. Although these objections are valid they are, on the other hand, often overstated in some instances. Some solution concepts may exhibit rather well those payoffs which are likely to be achieved without indicating the details of how they will actually be arrived at. Often, a

resulting payoff has implicit in it the coalition structure which would have formed in order to reach this outcome. E.g., for each imputation in a particular von Neumann-Morgenstern solution for a three-person game there is a natural way to associate a coalition structure with this outcome; except for a payoff interior to the core in which every coalition is satisfied simultaneously. It is also safe to say that more detailed insights about the social sciences have been obtained from the older static and side-payment models than from the more abstract dynamical or non side-payments theories which have come into existence in more recent years.

There now exists a fairly substantial amount of theory for games without side payments as well as for more dynamical approaches. There are also some interesting existence theorems for a few such models, even though several of these theories are somewhat deeper mathematically. Nevertheless, there is still a need for models which can exhibit more explicitly the dynamical, noncooperative, and coalitional formation aspects for the multiperson cooperative games. The purpose of this paper is to present a model which is extremely elementary in its basic definitions in order to pursue some of the more dynamical or negotiation aspects in greater depth and to hopefully arrive at better insights into such behavior. Eventually, such discoveries may be incorporated back into earlier models, as e.g. was the case when the nonexistence of solutions was first exhibited in generalizations of the von Neumann-Morgenstern theory before it was discovered for their classical theory (see section 5 in [25]).

In section 2 we briefly refer to some other attempts to incorporate dynamical, noncooperative, or coalitional structures more explicitly into cooperative games. Our simple model [27] which associates a unique payoff with each coalition structure is presented in section 3. The definitions for stable set and core are reviewed in section 4; and these solution concepts for one particular form of domination are described for all three-person games in the following section. A "real world" example, a three-person subgame of the ten-person "Communications Satellite Game," is solved in the final section for both the side-payment and non side-payment models.

## 2. Some Other Approaches.

One dynamical approach to cooperative games begins with an arbitary imputation and considers how the players may move successively, in discrete or continuous steps, through a sequence of such payoff vectors until they converge upon some outcome which is "stable," e.g., in the sense that it is in the kernel or core, is some type of known value, nucleolus or center, or is an equilibrium point of some sort. These "forces" or "transfer schemes" can be viewed intuitively as social pressures or bargaining steps moving towards an equitable or stable result. The set of all such limit points for a particular scheme can give rise to a new solution concept in the event that it does not correspond to a previously known one, e.g., as in the case of the lexicographic kernel of G. Kalai. A good deal of research in this direction has been done in the past ten years as is indicated in the papers by Scarf [41], Stearns [48], Billera [7], Wu [54], Wu and Billera [55], Grotte [15, 16], G. Kalai, Maschler and Owen [21], Owen [38], and Maschler and Peleg [30].

Several approaches to the cooperative games which began in the 1950s make use explicitly of the partitions of the set of n players (i.e., coalition structures), rather than just the subsets of players (i.e., coalitions). Such models include the theory of $\Psi$ stability of Luce [29], the bargaining sets of Aumann and Maschler [2] (consult Maschler for a detailed list of references), the games in partition function form introduced by Thrall (see [49]) and suggested also by Gamson [12]. Stable sets and cores were studied for this latter model by Lucas [22, 23, 24, 26, 49], and more recently by Fink [11] in a slightly different format. Shapley values have been investigated in this and more general contexts by Eisenman [9, 10], Gilbert [13], and Myerson [35, 36]. Aumann and Dreze [1] have also generalized the classical solution concepts to include partitions. Coalition structures have also been used in models developed by social scientists, and some references to this appear in Shenoy [46].

There have been suggestions by von Neumann (e.g., see page 25 in [53]), by Nash [37], and others to the effect that the negotiations and bargaining in a cooperative game should be considered as a noncooperative game superimposed on the cooperative structure. Work along these lines has been carried out by Vickrey [50], Harsanyi [18], Selten [43] and Weber [52]. Such approaches

can lead to using the normal form of a game and the theory of equilibrium points to get solution concepts for cooperative games. There are, however, some difficulties applying equilibrium points to "real world" problems (see e.g. [28]), as well as some interesting new and basic theoretical results for this concept [8]. The potential for using discoveries from the repeated play of games arises, and important new developments have also appeared in this area [6, 34]. The extensive form of a game can also be helpful in modeling negotiations; and even if there are a continuum of possible moves made continuously in time, then some of the recent work by J. G. Kljushin from Leningrad may prove useful.

Some new models for cooperative interactions have also been proposed by social scientists and the work by McKelvey and Ordeshook [32, 33] is an illustration of these developments.

Many of the models mentioned in this section do become quite abstract mathematically rather quickly and have not yet been very useful in applications. The object of this paper is to describe a highly simplified model with the hope that some of the structures discussed in this section can be applied to it without creating a theory which is technically intractable.

## 3. The Model.

We proceed to define an n-person cooperative game model which will be called the discrete partition function form. A unique n-dimensional vector payoff will be associated with each partition of $N$ into subsets. I.e., the outcomes for the individual players depend only upon which coalition structure actually forms. This can be viewed as the former model for games in partition function form [26, 49] except that side payments are not allowed.

Let $N = \{1, 2, \ldots, n\}$ be a set of $n$ players who are represented by 1, 2, ..., n. Let

$$P = \{P_1, P_2, \ldots, P_m\}$$

be an arbitrary partition of $N$ in nonempty and nonoverlapping subsets $P_1, P_2, \ldots, P_m$. A nonempty subset of $N$ is called a coalition and $P$ is referred to as a coalition structure. Denote the set of all partitions of $N$ by

$$\Pi = \{P\}.$$

Also denote the real numbers by $R$.

For each partition $P$ assume that there is an outcome function

$$F_P\colon\ N \to R$$

which assigns the real-number outcome or payoff $F_P(i)$ to each player $i$ when the partition $P$ is the one to form. The function

$$F\colon \Pi \to \{F_P\colon P \in \Pi\}$$

which assigns to each partition its outcome function is referred to as a discrete partition function. The ordered pair

$$(N,F)$$

is called an n-person game in discrete partition function form.

For each player $i$ in $N$ define the value of $i$ as

$$v(i) = \min_{\{P\in\Pi\colon\ \{i\}\in P\}} F_P(i).$$

This value $v(i)$ is the worst that can happen to $i$ considering all possible ways the players in $N - \{i\}$ can form into coalitions.

In summary, a discrete partition function game merely assigns a particular payoff vector

$$x^P = (x_1^P, x_2^P, \ldots, x_n^P) = (F_P(1), F_P(2), \ldots, F_P(n)) = F_P(N)$$

whenever the partition $P$ is the one which is actually realized as a result of playing the game. Player $i$ receives the amount $x_i^P$ when $P$ forms. The set of vectors $F_P(N)$, one for each $P \in \Pi$, is referred to as the set of extended imputations and is denoted by $E$. This set $E$ is a finite set in contrast to most models for multiperson cooperative games. Note that we have not assumed any "superadditivity" on the functions $F_P$ so that the points in $E$ are arbitrary. An imputation $x^P = F_P(N)$ is individually rational (i.r.) whenever

$$x_i^P \geq v(i) \quad \text{for all } i \in N.$$

The set of (i.r.) imputations will be denoted by

$$A = \{x^P \in E: x^P \geq v\}$$

where

$$v = (v(1), v(2), \ldots, v(n)).$$

For vectors such as $x$ and $y \in E$ or for $v$ above, we write $x \geq v$ for $x_i \geq v(i)$ for all $i \in N$, $x >_M y$ to mean $x_i > y_i$ for each $i \in M$, and similar expressions for $>$ and $\geq_M$.

## 4. Solution Concepts.

The set $A$ consists of all realizable and i.r. outcomes $x^P$ for the game $(N,F)$ and can be viewed as a "presolution" to this game. The problem is to determine which such outcomes are most likely to occur in the actual play of the game. The resulting payoffs may be based upon different concepts such as stability, bargaining power, equity, and so forth. In the theory of multiperson "cooperative" games these concepts take the form of the various "solution concepts" such as the von Neumann-Morgenstern solutions (stable sets), the core introduced by Gillies [14] and Shapley, the value concepts of Shapley [44] and others, the various bargaining sets of Aumann and Maschler [2] and others, the nucleolus of Schmeidler [42] and its variants, the subsolutions of Roth [40], etc. In this section we will describe models for stable sets and the core for the games $(N,F)$.

von Neumann and Morgenstern (vN-M) introduced the relation of "domination" on their form of the imputation set. Several variants of their definition have since appeared (e.g., see Fink [11]). We will now introduce five different types of domination relations between elements in our set $A$. Let $x^P$ and $y \in A$, let $P$ and $Q \in \Pi$, and let $M$ represent a nonempty subset of $N$. We will write

$$x^P \operatorname{dom}_M^r y$$

to mean that $x^P$ <u>dominates</u> $y$ <u>via</u> $M$ and that the domination is of type $r = 1,2,3,4$ or $5$. These five different types of domination are defined as follows.

(1) $x^P \operatorname{dom}_M^1 y \iff x^P >_M y$.

(2) $x^P \operatorname{dom}_M^2 y \iff x^P >_M y$ and $M = \bigcup_{i=1}^{q} P_i$ for $P_1,\ldots,P_q \in P$.

(3) $x^P \operatorname{dom}_M^3 y \iff x^P >_M y$ and $M \in P$.

(4) $x^P \operatorname{dom}_M^4 y \iff x^P >_M y$, $M \in P$, and $x^Q >_M y$ for all $Q \ni M$.

(5) $x^P \operatorname{dom}_M^5 y \iff x^P >_M y$, $M \in P$, and $x^Q \geq_M x^P$ for all $Q \ni M$.

Note that domination via $M$ of type $r$ implies that of type $r-1$ for $r = 2,3,4$ and 5.

We will also say that $x^P$ dominates $y$ through type $r$ domination, denoted

$$x^P \operatorname{dom}^r y$$

if there exists some such $M$ so that $x^P \operatorname{dom}_M^r y$. Furthermore, if $x \in A$ and $B \subset A$ we let

$$\operatorname{Dom}_M^r x = \{y \in A\colon x \operatorname{dom}_M^r y\}$$

$$\operatorname{Dom}^r x = \{y \in A\colon x \operatorname{dom}^r y\}$$

$$\operatorname{Dom}_M^r B = \{y \in A\colon x \operatorname{dom}_M^r y \text{ for some } x \in B\}$$

and

$$\operatorname{Dom}^r B = \{y \in A\colon x \operatorname{dom}^r y \text{ for some } x \in B\}$$

for $r = 1,2,3,4$ or 5.

In the remainder of this section we will delete the superscript $r$ from $\operatorname{dom}^r$ and $\operatorname{Dom}^r$ since the following definitions could be stated for each one of these five types of domination. In sections 5 and 6 the analysis is done only for domination of type 5, and thus dom and Dom

will stand for $\text{dom}^5$ and $\text{Dom}^5$ respectively in these final two sections. Similar investigations can be done for the other four types of domination.

A stable set (or vN-M solution) for (N,F), or for any pair such as (A, dom), is a set $V$ such that

$$V \cap \text{Dom } V = \emptyset,$$

where $\emptyset$ is the empty set, and

$$V \cup \text{Dom } V = A.$$

In other words,

$$V = A - \text{Dom } V$$

i.e., $V$ is fixed under the mapping

$$f\colon\ 2^N \to 2^N \quad \text{where} \quad f(B) = A - \text{Dom } B.$$

The core $C$ for the game $(N,F)$ consists of those imputations which are maximal with respect to the "dom" relation, i.e.,

$$C = A - \text{Dom } A.$$

For any model and solution concept such as $V$ or $C$, one is interested in questions about the existence, uniqueness, mathematical nature, computability, as well as applicability of these sets or ideas. The stable sets and the core for all three-person games in discrete partition function form are described explicity in the next section for type 5 domination.

The pair (A, dom) is a special case of an abstract game (vN-M [51]), i.e., an arbitrary set $A$ with a binary relation "dom" on this set. We can also view our particular case as a finite directed graph with node set $A$ and with an arc from $x$ to $y$ whenever $x$ dominates $y$ (or for some purposes when $x$ is dominated by $y$). A great deal is known about abstract games and such results appear in both the game theory and graph theory literature. A sample of such work appears in publications by Berge [5], Richardson [39], Harary and Richardson [17], Behzad and Harary [3,4], Roth [40], Shmadich [47], E. Kalai, Pazner, and Schmeidler [19,20], and Shenoy [46]. Additional references

appear in the report by Shenoy. Results of this type apply immediately to our model for games $G = (N,F)$.

The main reason, however, for introducing the simplified model $(N,F)$ is an attempt to gain greater insight into the dynamics of coalition formation Some previous attempts in this direction have been rather intractable mathematically. Our plan is to employ stochastic and dynamical techniques to this model in order to analyze the formation and breakup of coalitions, e.g., by using methods similar to Chapters 2 and 3 in Shenoy [46]. Some aspects of bargaining and negotiation can be introduced by analyzing this cooperative model in a noncooperative mode, making use of ideas from the normal and extensive forms of a game as well as from various models for repeated play of games, as was indicated in section 2. The elementary nature of the model $(N,F)$ has made the analysis of the noncooperative models built upon it more manageable, at least for small values of $n$. Some preliminary results in this direction have been obtained and will be reported elsewhere.

5. <u>The Three-person Games</u>.

For the case $n = 3$ we have $N = \{1, 2, 3\}$ and we get five partitions in the set $\Pi$:

$$P^1 = \{\{1\}, \{2,3\}\},\ P^2 = \{\{2\}, \{1,3\}\},\ P^3 = \{\{3\}, \{1,2\}\},$$
$$P^0 = \{\{1\}, \{2\}, \{3\}\}, \text{ and } P^N = \{N\}.$$

The set $E$ of extended imputations consists of the five corresponding imputations:

$$x^1 = (x_1^1, x_2^1, x_3^1),\ x^2 = (x_1^2, x_2^2, x_3^2),\ x^3 = (x_1^3, x_2^3, x_3^3),$$
$$x^0 = (x_1^0, x_2^0, x_3^0), \text{ and } x^N = (x_1^N, x_2^N, x_3^N).$$

The set $A$ of (i.r.) imputations is given by

$$A = \{x^\ell \in E\colon x^\ell \geq v\} \quad (\ell = 0, 1, 2, 3, \text{ and } N)$$

where $v = (v(1), v(2), v(3))$ and for each $i \in N$

$$v(i) = \min \{x_i^i, x_i^0\}.$$

In this section we will let $i$, $j$ and $k$ represent any permutation of the three <u>distinct</u> players 1, 2 and 3 in $N$.

We will first make some general remarks about domination (type 5) before exhibiting the stable sets $V$ and core $C$ for $A$ in the case when $n = 3$.

(1) For any $x \in A$, $x \notin \mathrm{Dom}_{\{i\}} A$ for all $i \in N$.

As a consequence of (1) we can find a stable set $V^0$ for $A - \{x^0\}$ and then merely check whether or not $x^0$ needs to be added to $V^0$ to obtain the desired stable set $V$ for $A$, i.e., whether $x^0 \notin \mathrm{Dom}\, V^0$ or $x^0 \in \mathrm{Dom}\, V^0$.

(2) If $x \in A$ and $x^N > x$, then $x \notin V$ for any stable set $V$.

As a result, we only need to determine a stable set $V^N$ for $A - (\{x^0, x^N\} \cup \mathrm{Dom}\, x^N)$ and then check whether or not $x^N$ needs to be added to $V^N$ to determine a stable set $V^0$ for $A - \{x^0\}$.

(3) In the case $n = 3$, if $x \in A$ and $x \text{ dom } y$, then $y$ cannot dominate $x$.

I.e., if $x$ is greater than $y$ on two or three components, then $y$ cannot

dominate $x$ because of (1).

(4) In the case $n = 3$, $x^i$ is <u>effective</u> for the coalition $\{j,k\}$ for each $i \in N$, i.e., $x^i \preceq_{\{j,k\}} x^Q$ for all $Q \ni \{j,k\}$. This follows from the fact that the two-person coalition $\{j,k\}$ appears in only the one partition $P^i$ corresponding to $x^i$. Consequently, the determination of the stable sets or core for $A$ in any three-person game depends to a large extent upon the donimation pattern between the three imputations $x^i$ and how these relate to $x^N$.

To determine all stable sets $V$ for the case $n = 3$ we consider the following four cases.

<u>Case</u> I. Assume that $x^i$, $x^j$ and $x^k \in \text{Dom } x^N$.

Then $V^0 = \{x^N\}$ if $x^N \in A$ and $V^0 = \emptyset$ otherwise. It follows from (1) that $V = V^0$ or $V^0 \cup \{x^0\}$ depending upon whether or not $x^0 \in \text{Dom } x^N$. We note that $V$ cannot be the empty set since $x^0 \in \text{Dom } x^N$ implies $x^N \in A$.

<u>Case</u> II. Assume that $x^i$ and $x^j \in \text{Dom } x^N$ and $x^k \notin \text{Dom } x^N$.

It follows that $V^N = \{x^k\}$ or $\emptyset$ depending upon whether or not $x^k \in A$. To obtain $V^0$ we use (2), i.e., we must add $x^N$ to $V^N$ iff $x^N \in A$ and $x^N \notin \text{Dom } V^N$. To arrive at $V$ from $V^0$ we then use (1), i.e., we must add $x^0$ to $V^0$ iff $x^0 \notin \text{Dom } V^0$.

<u>Case</u> III. Assume that $x^i \in \text{Dom } x^N$ and $x^j$ and $x^k \notin \text{Dom } x^N$.

(i) If there is no domination between $x^j$ and $x^k$ then $V^N = \{x^j, x^k\} \cap A$. We must add $x^N$ to $V^N$ to obtain $V^0$ iff $x^N \in A$ and $x^N \notin \text{Dom } V^N$. And we must add $x^0$ to $V^0$ to reach $V$ iff $x^0 \notin \text{Dom } V^0$.

(ii) Assume $x^j$ dom $x^k$. Then $V^N = \{x^j\} \cap A$ and $V^0$ and $V$ are obtained as in case (i) using (2) and (1) respectively.

<u>Case</u> IV. Assume that $x^i$, $x^j$ and $x^k \notin \text{Dom } x^N$.

(i) If there is no domination between $x^i$, $x^j$ and $x^k$ then

$V^N = \{x^i, x^j, x^k\} \cap A$, and $V^0$ and $V$ are obtained as above using (2) and (1).

(ii) If the only domination within the set

$$A' = \{x^i, x^j, x^k\}$$

is $x^i$ dom $x^j$, then $V^N = \{x^i, x^k\} \cap A$ and $V$ is obtained as before.

(iii) If the only domination within $A'$ is $x^i$ dom $x^j$ and $x^i$ dom $x^k$, then $V^N = \{x^i\} \cap A$ and $V$ is obtained as before.

(iv) If the only domination in $A'$ consists of $x^i$ dom $x^j$, $x^i$ dom $x^k$ and $x^j$ dom $x^k$, then $V$ is determined as in case (iii).

(v) If the only domination in $A'$ consists of $x^i$ dom $x^k$ and $x^j$ dom $x^k$, then $V^N = \{x^i, x^j\} \cap A$ and $V$ is obtained from $V^N$ using (2) and (1).

(vi) If the domination pattern in $A'$ is

$$x^i \text{ dom } x^j \text{ dom } x^k \text{ dom } x^i$$

then _no_ stable set $V$ exists for the game. This domination pattern implies that $x^i$, $x^j$ and $x^k \in A$.

Note that there exists a _unique_ nonempty stable set for every three-person game, except for our last Case IV, (vi) in which no such set exists. If our components $x_i^P$ were chosen at random from the unit interval, then the probability of nonexistence can be computed and it is less than one percent. However, in "real" applications it could be expected to occur more often.

It is a routine task to determine for a given game which imputations in $V$ are in Dom A, and to thus determine the core $C = A - \text{Dom } A = V - \text{Dom } A$ (when $V$ exists). In Case IV, (vi), in which $V$ does not exist, the core $C$ may or may not be the empty set depending upon whether or not $x^0$ and $x^N \in$ Dom A. More generally, algorithms for determining C for finite A are given in some of the references mentioned in section 4.

6. An Example.

In chapter 11 of his recent book [31], McDonald describes a business game concerning which American corporations would put up domestic communication satellites. Between 1960 and the mid 1970s this idea was conceived, became economically feasible, and was finally realized. This new technology gave rise to may potential benefits and could have drastically altered the rather stable and placid telecommunications industry. Some firms had the necessary technology and others had the required "traffic", and thus there were potential gains from cooperation, even between companies that had competed to some extent. In the late 1960s there were ten corporate groups, or players:

> AT&T, Comsat, Hughes, Western Union, General Telephone (GT&E), the Networks (ABC,CBS,NBC), RCA, MCI Lockheed, Western Tel, and Fairchild,

plus one nonstrategic "player" or "rule-maker" (the FCC). McDonald did not solve the full ten-person game, but he did provide the values for a particularly active three-person subgame which took place between General Telephone (G), Hughes (H) and Western Union (W). This subgame is discussed in detail in his book.

His value estimates are very crude; and they do not represent merely monetary consideration, but include many benefits which are difficult to quantify such as corporate image and their position in future business or "technological" games. Nevertheless, these rather vague numerical estimates were about the best that the participants themselves could do, and these numbers were checked with some corporation experts who were closely involved with the actual decision-making. McDonald's estimates for the three-person game with player set

$$N = \{G, H, W\}$$

were presented in discrete partition function form as follows:

$$F_{(G)(H)(W)}(N) = (1, 2, 3) \quad = x^0$$

$$F_{(G)(HW)}(N) \quad = (1, 4, 4) \quad = x^1$$

$$F_{(H)(GW)}(N) \quad = (1.5, 2, 5) = x^2$$

$$F_{(W)(GH)}(N) \quad = (4, 4.2, 3) = x^3$$

$$F_{(N)}(N) \qquad = (1, 2, 4) \quad = x^N$$

where (G)(HW) represents the partition $\{\{G\},\{H,W\}\}$, etc. These five vectors form the set A of imputations, and both the core and the unique stable set consist of the one payoff vector $x^3$ corresponding to the coalition structure $\{\{G,H\},\{W\}\}$. In fact, G and H did enter into a coalition and petitioned the FCC for a license to jointly orbit a satellite; whereas W desired to go it alone and has since put up its own bird. Even when the FCC suggested that they form the coalition N to protect W's risk, the three firms soon returned to the FCC with a slightly modified plan involving this same coalition structure.

The analysis in McDonald's book made use instead of the previous model for games in partition function form (with side payments) as described in [49]. Assuming additivity, the partition function (which assigns values to the respective coalitions in a partition) is given as follows. Here we have also "normalized" by subtracting off the values 1, 2 and 3 which the respective players can obtain by themselves.

$$F_{(G)(H)(W)} = (0, 0, 0)$$

$$F_{(G)(HW)} \quad = (0, 3)$$

$$F_{(H)(GW)} \quad = (0, 2.5)$$

$$F_{(W)(GH)} \quad = (0, 5.2)$$

$$F_{(N)} \qquad = (1).$$

In this approach the pareto-optimal part of the set of imputations is

the triangle

$$A_{(GH)(W)} = \{(x_G, x_H, x_W): \ x_G + x_H + x_W = 5.2 \text{ and } x_G, x_H, x_W \geq 0\}.$$

The core is empty, but "just barely" in the sense that a slight change of 0.3 in some of the values in the "right" direction could create a nonempty core.

The game has many stable sets, but all of them contain some imputations in the small triangle in $A_{(GH)(W)}$ with vertices (2.2, 3, 0), (2.2, 2.7, 0.3) and (2.5, 2.7, 0). And one would expect that the final outcome would be selected from the imputations near this region. So G and H may split the total amount 5.2 so that the latter obtains 2.7 to 3. However, there are theoretical arguments to support the allocation of a small side payment from G and H to W. In fact, this is what did occur in the real game. When the FCC questioned the coalition structure {{G,H}, {W}} and recommended {N} because of an element of risk if W were to go it alone, then H offered some of its technical information (i.e., the side payment) which would lower the risk to W (and hence its customers or stockholders who would pay for any such failure). So this rather crude side-payment model did (after the fact but in ignorance of it) suggest the small side payment which was actually realized in the real game.

More details about this game are given in McDonald's chapter 11.

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A new model for multiperson cooperative games in discrete partition function form is presented. A unique outcome vector is assigned to each partition of the set of n players into coalitions. Various domination relations and solution concepts are described on this set of outcome vectors. Some three-person examples are analyzed in detail.

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